A Methodological Framework for Optimizing Bus Transit Service Coverage

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ABSTRACT

A methodological framework is presented for determining the optimal length of transit routes that extend radially from the central business district (CBD) into low density suburbs. The problem of optimal route length may not be considered independently of route location and service scheduling. Therefore, the problem considered in this paper is finding an optimal combination of route length, route spacing, headway and fare which maximize either operator profit or social welfare for an urban corridor with elastic demand. The social welfare objective is optimized with both unconstrained subsidy and break-even constraints. The equations for the optimal design variables which maximize operator profit and social welfare are derived analytically for many-to-one travel patterns. The equations provide considerable insight into the optimality conditions and interrelations among variables. These equations are also incorporated within an efficient algorithm which computes the optimal values of decision variables for a more realistic model with vehicle capacity constraints. The algorithm is applied to a rectangular urban corridor with uniform passenger travel density. The numerical results show that at the optimum, the total operator profit and welfare functions are rather shallow, thus facilitating the tailoring of design variables to the actual street network and particular operating schedule without substantial decreases in profit and/or welfare. The social welfare function is relatively flat near the optimum for a relatively large range of subsidies. This result implies that for a given set of input data the break-even constraint may be an economically and politically preferable objective because it eliminates subsidy, while it reduces social welfare only marginally. The sensitivity of design variables to some important exogenous factors is also presented. The methodology presented in the paper is also applicable to the problem of optimal service coverage of feeder bus systems serving rapid rail line stations.

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INTRODUCTION

One of the main problems in designing bus transit services is to provide appropriate service coverage, and particularly to determine how far outward to extend transit routes into low density suburbs. Service operators and users have somewhat conflicting objectives regarding transit route lengths. Operators prefer short routes in order to minimize their costs. Passengers, especially those from the outer suburbs, prefer longer routes in order to minimize their access impedance. In cases where the demand for transit service is elastic (i.e., the passengers are sensitive to the level of service (LOS) characteristics and fare offered), shorter routes, and thus higher access impedance, might decrease service attractiveness and cause potential travelers to switch to other travel modes. Since the route length has a significant impact on both operator costs and passenger impedance, its value should be carefully selected.

The purpose of this paper is to develop a methodological framework for optimizing the length of bus transit routes that extend radially outward from the central business district (CBD), or those of a feeder bus system serving rapid rail line stations. However, this problem may not be considered independently of route location and service scheduling. Therefore, the problem considered here is finding an optimal combination of route length, route spacing, headway and fare which maximizes operator profit and social welfare for a rectangular-shaped urban corridor wherein passenger trip densities are uniformly distributed over the corridor.

In this paper, demand for transit service is elastic, meaning that service characteristics affect the ridership which in turn impacts earned revenue. Ridership also affects service characteristics and thus operator cost, since a certain level of service will have to be provided to attract a certain level of demand. The framework proposed in this paper recognizes these interactions between demand and supply (operator cost) and calculates equilibrium LOS characteristics and fares that optimize transit service coverage under several design objectives. The design objectives considered are: (1) maximization of operator profit and (2) maximization of social welfare with unconstrained subsidy and a (3) maximization of social welfare with break-even constraint. These objectives were optimized with and without the vehicle capacity constraint.

LITERATURE REVIEW

Several previous studies sought to optimize various elements of transit network design and service using calculus and, to a lesser extent, mathematical programming methods (1-22). An extensive review of optimization models can be found in Chang and Schonfeld (20). The summary of pertinent analytical models that are classified according
to the design variable(s) optimized is presented in Table 1. The table shows that in most studies, travel demand was inelastic and uniformly distributed over the service area. The usual travel pattern was many-to-one, which is typical for suburb to CBD commuting. The most common objective function was minimization of the sum of operator cost and user time cost. Obviously, the assumptions of inelastic demand precluded the models from analyzing impacts of pricing policies and subsidy on the system design characteristics and service coverage.

A paper by Kocur and Hendrickson (12) pioneered the development of an analytical model with elastic demand, and derived closed form solutions for optimal route spacing, headway and fare for different design objectives. The model offered interesting insights about the interrelationships among decision variables.

A literature review revealed only two published papers (15-16) that dealt with the optimization of a radial transit route length in an urban transportation corridor, which is the focus of this paper. Given the significant impact of the route length on cost, it is rather surprising that this research topic has not been given more attention in the literature. Wirasinghe and Seneviratne (15) developed closed form solutions for the rail transit line length for sectorial and rectangular corridors currently served by bus. The demand density was inelastic and uniformly distributed. The objective function to be minimized included the total cost of rail fleet, rail and bus operating cost and passenger time cost. Spasovic and Schonfeld (16) presented a model for optimal service coverage in the case of a rectangular and sectorial urban corridor with uniform and linearly decreasing density functions, that were inelastic. The model did not only optimize route length, but also jointly optimized headway, route and stop spacing. In contrast to the Wirasinghe and Seneviratne (15), this paper considered stations along the line and related access cost. Furthermore, the paper relaxed the Wirasinghe and Seneviratne assumption that during a peak-period, the route would operate at the maximum allowable headway. This assumption may be unwarranted even for the peak periods, since the optimal headway may be heavily influenced by user waiting time.

This paper extends the methodology of Spasovic and Schonfeld (16) to the case of a rectangular corridor with elastic demand. The assumption of elastic demand enables the proposed model to not only optimize the route length, but also jointly optimize the headway, route spacing and fare, thus analyzing impacts of pricing policies and subsidy on the system's design and service coverage.
METHODOLOGICAL FRAMEWORK

The methodological framework for planning optimal bus transit service coverage in which the resources and costs of providing the service are related to its operating characteristics and the induced demand from the service is presented in Figure 1. In this process, the values of the service characteristics such as route length, route spacing (or route density), headway (or its inverse, the frequency) and fare must be carefully selected to satisfy prespecified design objectives.

Since the demand for transit is elastic, the service characteristics chosen will have an impact on ridership, and thus, system revenue. As more frequent service is provided, more passengers will use it, thus generating higher revenues via the payment of fares. On the other hand, ridership will impact the service characteristics and thus operator cost because a certain number of buses must be operated on a route at a certain frequency in order to provide sufficient capacity to accommodate the induced demand. This in turn will increase operating costs.

The LOS characteristics and fare to be provided could be optimized using several objectives. For example, the maximization of operator profit -- defined as the difference between the fare box revenue and operating cost -- could be one of the design objectives. However, in the real world, the majority of transit systems do not recover the operating cost from the fare box and need to be subsidized from additional external revenue sources.

As was mentioned before, often there is a conflict between the operator's and users' objectives. The users would prefer to have short access to the route (e.g., frequent stops and longer routes) and little waiting time (i.e., frequent service). On the other hand, the operator would prefer to have very long headway and shorter, sparsely located routes with few stops, so that cost is minimized. In order to alleviate the perceived conflict between user and operator objectives, the sum of operator and user cost is often used as a suitable design criterion. The operator cost is self-explanatory, while the user cost is defined as the cost of time the user spends using the service (i.e., access, wait, in-vehicle riding, and access times multiplied by the perceived value of user time). In case of elastic demand with no requirements on minimum service characteristics to be provided, it is possible to find a set of LOS and fare that minimizes operator and user cost by effectively eliminating the transit demand (i.e., by driving the transit share of potential demand to zero). In this case the objective of minimizing the total system cost should be replaced with the maximization of social welfare (defined as a sum of consumer surplus and operator surplus -- profit) subject to a budget constraint.
In summary, the optimal bus service coverage from Figure 1 could be formulated as an optimization problem wherein the levels of various service characteristics must be chosen to improve (e.g., minimize or maximize) a certain objective in the form of service design functions. In this paper, the framework from Figure 1 is used to determine the optimal service coverage of a simplified bus transit system under various objectives. The resulting optimal service designs for each objective are compared to gain insights about when one objective is more appropriate than the others.

STUDY APPROACH

The problem under consideration is to provide optimal transit service coverage in an urban corridor as shown in Figure 2. The corridor of length $E$ and width $Y$ is divided into two zones. Zone 2 is the area between the CBD and the route terminus, while Zone 1 is the area between the route terminus and the end of the corridor.

The basic approach of this paper is to develop objective functions in which the various operator and user components are formulated as functions of several decision variables, such as route length, headway, route spacing and fare. The design objectives in determining the optimal service area coverage are maximization of the operator profit and maximization of social welfare. The optimal values of the decision variables are found by taking partial derivatives of the objective function of all decision variables, setting them equal to zero and solving them simultaneously. This approach, as it will be seen later, resulted in a simple model that offered considerable insight into the optimality conditions and interrelations among variables. The equations obtained are incorporated within an efficient algorithm which determines decision variable values for a more realistic model that includes a vehicle capacity constraint.

SYSTEM ASSUMPTION

A brief description of the modeled system follows.

**Bus System Characteristics**

1. An urban rectangular corridor is served by a bus transit system consisting of $n$ parallel routes of uniform length $L$, separated by a lateral spacing $M$.
2. The routes extend from the CBD outward.
3. The total transit demand is uniformly distributed along the entire corridor and over time, and is sensitive to the quality of transit service and fare.
4. The commuter travel pattern consists of many-to-one or one-to-many trips focused on the CBD.
5. There is a very dense rectangular grid street network that allows passengers orthogonal access movements (i.e., access paths are parallel and perpendicular to the route).
6. Transit vehicles operate in local service, (i.e., all vehicles serve all stations).
7. The average access speed is constant. Walking is assumed to be the only access mode.
8. Average wait time is assumed to equal one half the headway. The headway is uniform along the route, as well as among all parallel routes.
9. Operator costs are limited to those for vehicles (i.e., infrastructure is freely available).
10. There is no limit on vehicle fleet size.

Demand Functions

In this paper, the urban corridor demand is assumed to be a linear function sensitive to price and various travel time components (wait, access and in-vehicle time). A conceptual form of the demand density function is as follows:

\[ q = P \left[ 1 - e_w \cdot \text{wait time} - e_a \cdot \text{access time} - e_{iv} \cdot \text{in-vehicle time} - e_p \cdot \text{fare} \right] \]  \hspace{1cm} (1)

where:
- \( q \) = unit transit demand density \([\text{passengers/mile}^2\cdot\text{hour}]\)
- \( P \) = potential travel demand density \([\text{passengers/mile}^2\cdot\text{hour}]\)
- \( e_w \) = sensitivity factor for wait time
- \( e_a \) = sensitivity factor for access time
- \( e_{iv} \) = sensitivity factor for in-vehicle time
- \( e_p \) = sensitivity factor for fare

The demand function is similar to the one suggested by Kocur and Hendrickson (12), and almost identical to that of Chang and Schonfeld (20).

For the particular application studied in this paper, the total demand consists of the sum of Zone 1 and Zone 2 demand. Since the trip origins are uniformly distributed over the corridor served by parallel routes, an average passenger accessing the route perpendicularly walks one quarter of the spacing between the two routes -- an access distance of \( M/4 \). The length of passenger access alongside the route depends on whether the trip originated within Zone 1 or Zone 2. Passengers originating in Zone 1 have no other choice but to board the vehicles at the terminus, thus having an average access distance of \((E-L)/2 + M/4\). A passenger from Zone 2 walks along the route \( 1/4 \) of the local stop spacing \( S \) before reaching the stop. Therefore, in Zone 1, the total access time for an average passenger equals the average access distance divided by the access speed \( g \) (i.e., \((E-L)/2g + M/4g\)). For a passenger in Zone 2 the access time is \((M+S)/4g\). It is obvious that access times will affect demand in Zones 1 and 2.
The in-vehicle time consists of the actual riding time between the stop of origin and the CBD. The average in-vehicle riding time is obtained as the average distance traveled divided by the speed $V$, and is different for each zone. Therefore, the passengers originating in Zone 1 travel the whole length of the route $L$, while those from Zone 2 travel approximately an average distance $L/2$. According to Assumption 8, the passengers will wait half of the headway, $H/2$.

The hourly transit demand in Zone 1, $Q_1$ is then given as:

$$Q_1 = P Y (E - L)(1 - e_w H/2 - e_a(M + E - L)/2 - e_a L/V - e_{sf})$$  \hspace{1cm} (2a)

The hourly transit demand in Zone 2, $Q_2$ is given as:

$$Q_2 = P Y L (1 - e_w H/2 - e_a M + S - e_a L/2V - e_{sf})$$  \hspace{1cm} (2b)

The total hourly transit demand, $Q$, is then given as:

$$Q = Q_1 + Q_2$$  \hspace{1cm} (3)

$Q_1$: transit demand in Zone 1  \hspace{1cm} [passengers/hr]

$Q_2$: transit demand in Zone 2  \hspace{1cm} [passengers/hr]

$Q$: total corridor transit demand  \hspace{1cm} [passengers/hr]

$g$: access speed  \hspace{1cm} [km/hour]

$E$: corridor length  \hspace{1cm} [km]

$H$: operating headway  \hspace{1cm} [hours/vehicle]

$L$: length of transit route  \hspace{1cm} [km]

$M$: lateral route spacing  \hspace{1cm} [km/route]

$P$: potential transit trip density  \hspace{1cm} [passengers/km²-hour]

$S$: average stop spacing  \hspace{1cm} [km/stop]

$V$: average transit operating speed  \hspace{1cm} [km/hour]

$Y$: corridor width  \hspace{1cm} [km]

**Operator Cost**

The operator cost includes maintenance and overhead as well as the more direct costs of operation (driver wage, fuel, brake shoes, etc.) Vehicle depreciation costs might also be included as a portion of operator cost. In this paper, the operator cost is defined via the hourly operating cost $c$. The total hourly operator cost is defined as the active fleet size multiplied by the hourly operating cost. By definition, the fleet size is the number of on-line vehicles required to provide service, and is obtained by dividing the total round trip time (running time and layover time) by the headway. The average transit operating speed is selected to reflect running and layover times. Therefore, the total round trip time
is the round trip route length divided by average speed. The total hourly operator cost is then:

\[ Co = \frac{2cYLM}{HMV} \]  

(4)

where:

- \( Co \) = operator cost \([$/hour]\)
- \( c \) = vehicle operating cost \([$/vehicle-hour]\)
- \( Y \) = corridor width \([km]\)
- \( L \) = length of transit route \([km/route]\)
- \( H \) = route headway \([hours/vehicle]\)
- \( M \) = route spacing \([km/route]\)
- \( V \) = average transit speed \([km/hour]\)

**TRANSIT SERVICE DESIGN OBJECTIVES**

Two objectives are considered in this paper: (1) maximizing operator profit and (2) maximizing social welfare. The motivation was to optimize service coverage under each objective, compare the results and derive insights about optimal coverage. The derivation of objectives is discussed below.

**Maximizing Operator Profit**

The operator profit \( \Pi \) is defined as a difference between the "farebox" revenue \( TR \) and operator cost \( Co \):

\[ \Pi = TR - Co \]  

(5)

The total revenue \( TR \) is defined as a fare multiplied by the total ridership:

\[ TR = PYE \left[ 1 - e_o \frac{H}{2} - e_a \frac{M}{4g} - e_{of} f \right] f + PYL \left( E - L \right) \left[ 1 - e_o \frac{E - L}{2g} - e_v \frac{L}{V} \right] f + PYL \left[ -e_o \frac{S}{4g} - e_v \frac{L}{2V} \right] f \]  

(6)

The operator cost \( Co \) has been defined in Eq 4. The hourly total operator profit \( \Pi \) is then defined as a difference between the total operator revenue (Eq. 6) and operator cost (Eq.4):

\[ P = PYE \left[ 1 - e_o \frac{H}{2} - e_a \frac{M}{4g} - e_{of} f \right] f + PYL \left( E - L \right) \left[ 1 - e_o \frac{E - L}{2g} - e_v \frac{L}{V} \right] f + PYL \left[ -e_o \frac{S}{4g} - e_v \frac{L}{2V} \right] f \]  

\[ - \frac{2cYLM}{HMV} \]  

(7)
The total profit function can be maximized by setting its partial derivatives with respect to the decision variables to zero. In this case, the partial derivatives of the decision variables, the route length $L$, headway $H$, route spacing $M$, and fare $f$ are as follows:

\[
\begin{align*}
\frac{P(L)}{L} &= -\frac{2cY}{HMV} + 2 \frac{PYE - L}{2g} (-1)(-e) f \\
- \frac{eYV}{V} + 2 \frac{PYLE \gamma f}{V} + PYf \left( -e \frac{S}{4g} - 2eL \right) &= 0 \quad (8a) \\
\frac{P(H)}{H} &= -PEY \epsilon f + \frac{2cYL}{H^2 MV} = 0 \quad (8b) \\
\frac{P(M)}{M} &= -PEY \epsilon f + \frac{2cYL}{HM^2 V} = 0 \quad (8c) \\
\frac{P(f)}{f} &= PY \left[ 1 - e \frac{H}{2} - e \frac{M}{4g} - 2e \right] \\
+ PY (E - L)(-e \frac{E - L}{2g} - eL) + PYL(-e \frac{S}{4g} - eL) &= 0 \quad (8d)
\end{align*}
\]

When Eqs. 8a-8d are solved independently, we obtain the following equations:

\[
\begin{align*}
L^* &= E - \frac{2cY}{PHMf (eV - eVg)} - \frac{eSV}{4(eV - eVg)} \quad (9a) \\
H^* &= \left[ \frac{4cL}{eYMEV} \right]^{\frac{1}{2}} \quad (9b) \\
M^* &= \left[ \frac{8cLg}{eHPEV} \right]^{\frac{1}{2}} \quad (9c) \\
f^* &= \frac{1}{2e} - \frac{eH}{4eE} - \frac{eS(ME + 2(E - L)^2 + SL)}{8eEg} - \frac{e(2LE - L^2)}{4eEg} \quad (9d)
\end{align*}
\]

Solving Eqs. (9b) and (9c) simultaneously, the following expressions for $H$ and $M$ are obtained:

\[
\begin{align*}
H^* &= \left[ \frac{2cLe}{PEV} \right]^{\frac{1}{3}} \quad (10b) \\
M^* &= \left[ \frac{16cLe g^2}{PEV} \right]^{\frac{1}{3}} \quad (10c)
\end{align*}
\]
Several observations should be made here: When the route length, route spacing, headway and fare are optimized independently of each other, their relation to the other decision variables can be read directly from Eqs. 9a-9d. These equations provide the optimal value of one of the decision variable as a function of the other three variables and provide useful insights into the relationship between the decision variables and the various parameters. For example, Eq. 9a can be used to find the optimal route length when the headway, route and average stop spacing are given (e.g., to satisfy the minimum service standards). Such equations may be useful by themselves in some situations where certain decision variables such as the route length \( L \) or the fare \( f \) cannot be modified. Unfortunately, Eqs. 9a-9d cannot be solved simultaneously using algebraic methods.

According to Eq. 9a, the optimal route length varies directly with the corridor length \( E \), passenger density \( P \), operating headway \( H \), route spacing \( M \), fare \( f \), sensitivity factor for access time \( e_a \), transit speed \( V \). It varies inversely with the vehicle operating cost \( c \), access speed \( g \), stop spacing \( S \), and sensitivity factor for in-vehicle time \( e_{iv} \).

The optimal headway varies directly with the cube root of operator cost \( c \), route length \( L \), and sensitivity factor \( e_a \). It varies inversely with the cube root of the sensitivity factor for wait time \( e_w \) squared, potential passenger density \( P \), transit speed \( V \), corridor length \( E \), fare \( f \) and access speed \( g \).

The optimal route spacing varies directly with the cube root of access speed \( g \) squared, operator cost \( c \), route length \( L \), and sensitivity factor for wait time \( e_w \). It varies inversely with the cube root of the sensitivity factor for access time \( e_a \) squared, passenger density \( P \), transit speed \( V \), corridor length \( E \), and fare \( f \).

Note that the simultaneous solution of Eqs. 10b and 10c yields a result that shows optimality of a constant ratio between route spacing and headway. The following relationship between the headway and route spacing holds:

\[
M^* = 2H^* \left( \frac{e_w g}{e_a} \right)
\]  
(10e)

This result is similar to that obtained by Kocur and Hendrickson (12), and Chang and Schonfeld (20).

Finally, the optimal fare varies directly with the access speed \( g \) and transit speed \( V \). It varies inversely with the fare sensitivity factor \( e_p \), corridor length \( E \), sensitivity factor for wait time \( e_w \), headway \( H \), sensitivity factor for access time \( e_a \), and route spacing \( M \).

Maximize Social Welfare
The social welfare $W$ is defined as a sum of the consumer surplus $T$ and producer surplus $\Pi$:

$$ W = T + \Pi $$  \hspace{1cm} (11)

The producer surplus, also known as profit, is defined in the previous section and given by Eq. (7).

The consumer surplus $T$ can be defined as the total social benefit minus the total cost that users actually pay (Kocur and Hendrickson (12)). The total social benefits (also known as the users’ willingness to pay) for each of the zones can be obtained by inverting the demand for Zones 1 and 2 (Eq. 2a and 2b) to find the fare as a function of demand, and by integrating the inverted functions over the demand. Then the total consumer surplus $T$ can be stated as:

$$ T = \frac{PY(E-L)}{2e_p} \left[ 1 - e_w \frac{H}{2} - e_a \left( \frac{M}{4g} + \frac{E-L}{2g} \right) - e_v \frac{L}{V} - e_{vf} \right]^2 + \frac{PYL}{2e_p} \left[ 1 - e_v \frac{H}{2} - e_a \frac{M+S}{4g} - e_v \frac{L}{2V} - e_{vf} \right]^2 $$  \hspace{1cm} (12)

Therefore, the social welfare objective can be formulated as follows:

$$ W = \frac{PY(E-L)}{2e_p} \left[ 1 - e_w \frac{H}{2} - e_a \left( \frac{M}{4g} + \frac{E-L}{2g} \right) - e_v \frac{L}{V} - e_{vf} \right]^2 + \frac{PYL}{2e_p} \left[ 1 - e_v \frac{H}{2} - e_a \frac{M+S}{4g} - e_v \frac{L}{2V} - e_{vf} \right]^2 + PYNL \left[ 1 - e_v \frac{H}{2} - e_a \frac{M+S}{4g} - e_v \frac{L}{2V} - e_{vf} \right] \left( 1 - \frac{2cYL}{HMV} \right) $$  \hspace{1cm} (13)

In solving for the maximization of social welfare, a deficit constraint is considered. This constraint states that the operator cost must be equal to the sum of the total revenue $TR$ and the pre-specified acceptable level of subsidy $K$. The constraint can be written in the following form:

$$ Co = TR + K $$  \hspace{1cm} (14)

In this paper, the deficit constraint is as follows:

$$ \frac{2cYL}{HMV} - PY(E-L)\left[ 1 - e_w \frac{H}{2} - e_a \left( \frac{M}{4g} + \frac{E-L}{2g} \right) - e_v \frac{L}{V} - e_{vf} \right] f - PYL \left[ 1 - e_v \frac{H}{2} - e_a \frac{M+S}{4g} - e_v \frac{L}{2V} - e_{vf} \right] f - K = 0 $$  \hspace{1cm} (15)

**Unconstrained Subsidy Results**

In the case with unconstrained subsidy, the first order conditions at optimum are:

$$ \frac{W}{L} = 0 $$  \hspace{1cm} (16)
The optimized fare can be immediately obtained for Eq. 19, and is \( f^* = 0 \).

This result is not surprising since the marginal operator cost is zero in the bus system used in this paper. A similar result is found in Kocur and Hendrickson (12) and Chang and Schonfeld (20). The marginal cost and thus fare would become positive if a vehicle capacity constraint is introduced, as it will be shown later in the analysis of numerical results.

By substituting the zero fare back into Eqs. 16-18, the expressions for the optimal route length, spacing and headway are obtained as follows (Appendices A, 1,3,4):

\[
L^* = \frac{[2(AB + CD) - A^2 E]}{3(A^2 - C^2)} - \frac{1}{2} [12 ce_p (A^2 - C^2) + HMVP [(AB + CD)^2 + 3(AD + BC)^2]] + EHMVP \left( A^4 E + 2 A^3 B - 4 A^2 CD - 6 ABC^2 \right) \frac{1}{2} / (C^2 - A^2) \sqrt{HMVP}
\]

\[
H^* = \frac{\sqrt{J}}{\sqrt{A + B(1 - C - D + F - 2G) + L(1 - G - I)}}
\]

\[
M^* = \frac{\sqrt{J}}{\sqrt{A + B(1 - C - D + F - 2G) + L(1 - G - I)}}
\]

**Results with Break-Even Constraint**

The expressions for the optimal route length, spacing and headway are obtained as follows (Appendices A, 2,3,4):

\[
L^* = R \cdot \frac{1}{2} \left[ S(1 + \lambda) + T + fHMVP e_p 2(A + B) \left[ B(A + 3C) + D(3A + C) \right] (1 + \lambda) + fEHMVP e_p 2 A(A - 3C)(A + C) + f^2 HMVP e_p^2 A(1 + \lambda)^2 \right] \frac{1}{2} / (C^2 - A^2) \sqrt{HMVP}
\]

\[
H^* = \frac{\sqrt{J(1 + \lambda)}}{\sqrt{A + B(1 - C - D + F - 2G) + L(1 - G - I) + e_{ef} (-B + (E - L) + E\lambda)}}
\]
\[ M^* = \frac{\sqrt{J(1 + \lambda)}}{\sqrt{A + B(1 - C - D + F - 2G) + L(1 - G - I) + e_{\theta f}(-B + (E - L) + E\lambda)}} \] (25)

\[ f^* = \frac{2g\lambda L e_{\theta}(1 - 2E + L) - 2e_{\theta} \lambda V(E - L)^2 + e_{\theta} \lambda V(\frac{L - E}{EM}) + 2Eg \lambda V(2 - He_{\theta})}{4Eg_{\theta}V(1 + 2\lambda)} \] (26)

And if:
\[ X = 4A - 4B + ICDJ + IFGJ \]
and
\[ Z = I^2 e_{\theta}(C + F) \]
then \( \lambda^* \), the shadow price for the deficit constraint is (Appendix A.5):

\[ \lambda^* = -\frac{X + \sqrt{X^2 - 4(A - B)(X + ICDJ + IFGJ - Z)}}{2(X + ICDJ + IFGJ - Z)} \] (27)

**Capacity Constrained Headway**

The analytic models presented so far have not taken into account a vehicle capacity constraint. This constraint ensures that the total capacity provided on the routes satisfies the demand by restricting the maximum allowable headway. The constraint is written as:

\[ PYE \left[1 - e_{\theta} \frac{H}{2} - e_{\theta} \frac{M}{4g} - e_{\theta f}\right] + PY(E - L)\left[-e_{\theta} \frac{E - L}{2g} - e_{\theta} \frac{L}{V}\right] \]

\[ + PLY\left[-e_{\theta} \frac{S}{4g} - e_{\theta} \frac{L}{2V}\right] \frac{K Y}{MH} l \] (28)

where \( K = \) capacity of transit vehicle (in spaces), and
\( l = \) allowable peak load factor (at the CBD).

This constraint determines the expression for maximum allowable headway which is used within an optimization algorithm that is described next.

**OPTIMIZATION ALGORITHM**

Although the models presented so far provided valuable insights into the relations among decision variables and exogenous parameters, they were still too complex for simultaneously optimizing all the decision variables algebraically. To solve the model, an algorithm was developed which sequentially applied Eqs. 9a-9d (or Eqs. 20-22 or 23-26 depending on the objective to be optimized) to advance from an initial feasible solution toward the optimal solution. The algorithm, shown in Figure 3, starts with a trivial feasible solution to the problem and in each step improves the value of the objective.
function by computing an optimal value of one decision variable while keeping the others at their feasible levels. In computing the optimal values of decision variables, the algorithm first computes the route length, route spacing, headway and finally fare. In each step, the value of a newly computed variable is recorded and used in the next step for computing the optimal values of the other decision variables. The algorithm keeps improving the objective function until it converges to an optimal solution. The algorithm terminates when the values of total costs from two successive iterations are sufficiently close that no significant further improvement can be expected. Assuming that the optimal set of decision variables is reached, the program computes the values of the objective function (e.g., for the optimal route spacing \( M^* \), headway \( H^* \), fare \( f^* \)) allowing variations in the length of line \( L \), while maintaining the feasibility of the solution. The purpose of this is to investigate the shape of the total profit or social welfare functions near the optimum. As discussed later, the objectives turned out to be relatively flat (shallow four-dimensional U-shaped) functions. Thus, small deviations from the optimal decision variables result in even smaller relative changes in the objective value.

It is quite possible that buses may overload if no capacity constraint is introduced. Instead of formulating a model as a constrained optimization problem with non-linear objective function and linear constraint and solving it by using a penalty method (23), the following modification of the algorithm is made to take into account the vehicle capacity constraint:

1. Examine whether the newly obtained optimal headway satisfies the capacity constraint, i.e., whether the optimal headway is smaller than the maximum allowable headway, by computing the optimal bus load and checking whether the bus load exceeds the capacity.
2. If the bus load is smaller than the available capacity, there is no need for capacity constrained results.
3. Otherwise, set the optimal headway equal to the maximum allowable headway that is obtained by solving Eq (28), which is as follows (Appendix A, Eq. 6):

\[
H^* = C^2D - E + AE + BE + Q + FL + GL
\]

\[
+ \sqrt{((-C^2D + E - AE - BE - Q - FL - GL)^2 + 4XZ)}
\]

\[
2Z
\]

Calculate the set of decision variables that satisfies the convergence constraint. The set is considered an optimal solution.

**NUMERICAL EXAMPLE**
A numerical example is developed and used to demonstrate the applicability of the proposed models in optimizing transit service coverage. The motivation was to compare the optimal service designs obtained for each of the objectives and gain insights about when one objective is more appropriate to be used than the other.

The results, the optimal route length, route spacing, operating headway, fare, operator profit and social welfare (and consumer surplus) are shown in Table 2. These results present the optimized service variables for a 8.04x4.824 km (5x3-mile) rectangular corridor, with a potential demand density of 77.35 persons/km$^2$-hour (200 persons/mile$^2$-hour). The bus hourly operating cost is assumed to be $40. The complete set of input data is given in Figure 4. Part a of Table 2 presents results of the maximization of operator profit objective and social welfare both with the unconstrained subsidy and break-even constraints. Since the bus loads exceeded the capacity of buses, the optimal bus service design with vehicle capacity constraints are computed and presented in part b of Table 2. The optimal bus transit systems for (1) profit maximization, (2) social welfare maximization with unconstrained subsidy, and (3) social welfare maximization with break-even constraint is shown in Figures 4-6, respectively.

Some observations about these numerical results presented separately for cases with and without vehicle capacity constraint are as follows.

**Results Without the Vehicle Capacity Constraint:**

(a) For profit maximizing objective, there is an optimal service design with the route length of 7.117 km (4.426 miles), route spacing of 2.502 km (1.556 miles), headway of 0.311 hours and $0.67 fare. These values of service variables induce an hourly ridership of 1,007 passengers yielding a $456 profit. The overcrowding of buses with an average load of 162.5 passengers per bus allows for longer routes.

(b) For welfare maximizing objective with unconstrained subsidy, there is a optimal service design with the route length of 7.16 km (4.453 miles), route spacing of 1.944 km (1.209 miles), headway of 0.242 hours and $0 fare. The induced hourly ridership is 2,161 passengers, and the social welfare of $1196 is achieved with the $365.711 in subsidy. The introduction of the break-even constraint results in the optimal service design with the route length of 7.122 km (4.429 miles), route spacing of 2.036 km (1.266 miles), headway of 0.253 hours and $0.177 fare. There is an hourly ridership of 1,869 passengers and the social welfare is optimized at $1,170.

(c) A comparison of welfare maximization results with the unconstrained subsidy and break even constraints reveals that when the break-even constraint is removed the welfare increases slightly (by $25.949 or approximately 2.2%), while the deficit increases far
more (from zero to $365.711). The welfare function appears to be relatively flat near the optimum; a minor deviations away from the optimum will not decrease welfare significantly. This implies that for given input data, the welfare objective with break-even constraint seems quite reasonable and far more desirable from an economical standpoint than the welfare objective with unconstrained subsidy.

(d) The shadow price, shown in part a of Table 2, implies that relaxing the break-even constraint and thus increasing the operator deficit from zero to one dollar (i.e., to a $1 subsidy) will result in a $0.166 increase in welfare.

(e) The equilibrium demand is strongly influenced by the level of service and by fares optimized under different objectives. The total hourly demand level is 33.5% of the potential demand for profit maximization. It is 72.04% and 62.31% for welfare maximization for unconstrained subsidy and break-even constraints, respectively.

Results With the Vehicle Capacity Constraint:

(f) For the profit maximizing objective with the vehicle capacity constraint, there is an optimal route length of 5.3 km (3.296 miles), route spacing of 1.614 km (1.004 miles), headway of 0.201 hours and $0.883 fare. The induced hourly ridership is 743 passengers, yielding a $393 profit.

(g) For welfare maximizing objective with capacity constraint and unconstrained subsidy, the optimal service design has the route length of 4.557 km (2.834 miles), spacing of 1.124 km (0.699 miles), headway of 0.140 hours and $0.356 fare. These values of service variables induce an hourly ridership of 1,536 passengers, and the social welfare of $719 is achieved at the $150 subsidy. For the welfare maximizing objective with break-even constraint, the optimal service design consists of a 4.564 km (2.838 mile) route length, the route spacing of 1.188 km (0.739 miles), headway of 0.148 hours and $0.454 fare. These service variables induce an hourly ridership of 1,372 passengers, yielding a social welfare of $710.525.

(h) A comparison of welfare maximization results with the unconstrained subsidy and break even constraints in case with capacity constraint reveals that when the break-even constraint is removed the welfare increases slightly (by $8.5 or approximately 1.2%), while the deficit increases far more (from zero to $150). This implies that for given input data, the welfare objective with break-even constraint seems quite reasonable and far more desirable from an economical standpoint than the welfare objective with unconstrained subsidy.

(i) The shadow price, shown in part b of Table 2, implies that an increase in subsidy of $1 will yield a $0.128 increase in welfare.
The equilibrium demand is strongly influenced by the level of service and fares optimized under different objectives. The total hourly demand level is 24.8% of the potential demand for profit maximization. It is 51.2% and 45.76% for welfare maximization for unconstrained subsidy and break-even constraints, respectively.

**Comparison of Results**

In cases without a vehicle capacity constraint, profit maximization yields a smaller route length than welfare maximization, while the opposite is true (profit maximization yields a larger route length than welfare maximization) for the capacity constrained cases. A comparison of the optimal route length for different objectives indicates that profit maximization yields longer routes of 5.29 km (3.29 miles) than welfare maximization (with lengths of 4.557 km (2.834 miles)). Also, the optimal design for profit maximization has a longer headway and route spacing than that for welfare maximization (i.e., 0.201 hrs and 1.614 km (1.004 miles) vs 0.1 hrs and 1.124 km (0.699 miles)). This result can be explained as follows. The presence of the vehicle capacity constraint and the customers higher value for wait and access times rather than for in-vehicle time (as indicated by the values of sensitivity coefficients for travel time components) cause that in order to maximize welfare (and thus demand), the transit system with longer routes is replaced by one with denser routes and more frequent service.

A comparison of results in parts a and b of Table 2 shows that for profit maximization, the route length without the vehicle capacity constraint is 34% longer with the vehicle capacity constraint. The absence of the capacity constraint allows for a very high bus load. Due to the large number of passengers per bus, the revenue increases substantially, while the operator cost is kept low because of the longer headway and route spacing. However, the overload is far beyond the acceptable bus load. When a vehicle capacity constraint is introduced the bus load dropped three-fold (from 162 to 50) causing the operator to reduce cost by shortening the routes. The operator profit would be increased by $5.753 if the bus capacity were increased by one. Similarly, for welfare maximization with unconstrained subsidy, the route length without the vehicle capacity constraint is 56% longer than that with the vehicle capacity constraint. When the vehicle capacity constraint is introduced the bus load dropped three-fold (from 210 to 50), reducing the cost by shortening the routes.

A comparison of welfare maximization results with the unconstrained subsidy and break even constraints without capacity constraint reveals that when the break-even constraint is removed the welfare increases slightly (by $26 or approximately 2%), while the deficit increases far more (from zero to $365). A comparison of welfare maximization
results with the unconstrained subsidy and break even constraints in case with capacity constraint reveals that when the break-even constraint is removed the welfare increases slightly (by $8.5 or approximately 1.2%), while the deficit increases far more (from zero to $150). The total cost function appears to be relatively flat near the optimum. This indicates that minor deviations away from the optimum will not decrease welfare significantly. This result is similar to the one found in Chang and Schonfeld (20). The result implies that for given input data, the welfare objective with break-even constraint seems quite reasonable and far more desirable from an economical standpoint than the welfare objective with unconstrained subsidy.

(n) The shadow prices, shown in Tables 2 and 3, imply that increasing the operator deficit from zero to one dollar (i.e., there is a $1 subsidy) will result in a $0.166 increase in the welfare for the case without a vehicle capacity constraint and by $0.128 for the case with a vehicle capacity constraint.

**Sensitivity Analysis**

The sensitivity analysis is performed to show how changes in the more important exogenous parameters affect the values of the decision variables and objective functions. The changes in design variables, namely route length, spacing, headway and fare with respect to the corridor length, passenger density, transit and access speed, operator cost and elasticity factors (i.e., values of access, waiting, in-vehicle times) and fare are shown in Table 3.

The table shows that, for example, if corridor length is increased by 10% (from 5 miles to 5.5 miles) the route length is increased by 10.4%. This implies that the optimal route length \( L \) is elastic (i.e., the absolute value of the elasticity exceeds 1.0) with respect to the corridor length \( E \). The reason for this is that as the length of the corridor \( E \) is increased, the length of the area between the terminus and the end of the corridor, \( E-L \), is increased very slowly, thus increasing \( L \) faster than \( E \). This result is consistent to those obtained in Spasovic and Schonfeld (16) for fixed demand systems. Also, if the passenger density is increased by 10% the headway will be reduced by 5%. This result confirms that headway varies inversely (approximately) with the cube root of the passenger density. Table 3 also shows that, the route length would decrease by 5% if the sensitivity factor for fare is increased by 10%.

Table 3 also shows the effect of the change in profit objective as a result of change in elasticity factors. The effect of the route length on profit is shown in Figure 7. For a given route length, the system design variables have been re-optimized yielding the optimal profit. The profit function is relatively flat near optimum. A practical application
of this result is that for a given set of data input the optimal design variables can be
tailored to the actual network without substantially reducing the optimal profit.

The results in Table 2 suggest that the break-even constraint may well be
preferable to the unconstrained welfare maximization since its removal results in a
marginal increase in welfare. (It should be recalled that when the vehicle capacity
constraint is binding, the welfare is increased by $8.5 while the deficit increases from
zero to $150). Next, the sensitivity of welfare to the sensitivity factors used in the demand
function is examined.

Table 4 shows the change in welfare as a result of change in sensitivity factors.
The effect of route length on welfare is shown in Figure 8. As in the case of profit
maximization, for given route length the system design variables have been re-optimized
yielding the optimal welfare. The welfare functions are relatively flat near optimum.

The effect of subsidy on welfare and consumer surplus are shown in part A of
Figure 9. For a given subsidy level, the system design variables have been re-optimized
yielding the optimal welfare for each service design. The consumer surplus increases with
the subsidy. For $0 subsidy, the break-even constraint holds, the social welfare equals
consumer surplus. The net effect of profit and consumer surplus interactions is that the
welfare function is relatively flat near optimum for relatively large range of subsidies, as
shown in part A of Figure 9. This result is similar to the one found in Chang and
Schonfeld (20). A practical implication of this result is that for a given input data the
break-even constraint may be economically and politically preferable because it
eliminates subsidy and marginally reduces social welfare. Furthermore, as shown in part
B Figure 9, negative subsidy (profit) can be obtained by marginally decreasing welfare.

CONCLUSIONS

The paper presented a model of optimal bus transit service coverage that was
optimized to maximize profit and social welfare with unconstrained subsidy and break-
even constraint. The model provides simple guidelines for optimizing the extent of transit
routes and other major operating characteristics such as route spacing, headway and fare.
Eqs. 9a-9d. can be used to separately optimize route length, route spacing, headway and
stop spacing. They provide insights in interrelationships among the optimized variables.
For example, the cube root in Eqs. 9bb-9cc indicates that optimal solutions for headway
and route spacing are relatively insensitive to changes in system parameters.

The optimality of a constant ratio between route spacing and headway, which has
been found in previous studies for various bus network and demand conditions, for
example in Kocur and Hendrickson (12), and Chang and Schonfeld (20), is also found to
be maintained in the optimal solution of our model which in addition to the variables maximized by those authors optimizes the route length as well. The equations are incorporated within an efficient algorithm that provides an accurate method for simultaneously optimizing the decision variables.

Worth noting are the interrelationships among the optimization results for various optimization objectives. For example, the route spacing and headway that optimize profit, welfare with unconstrained subsidy, and welfare with break even constraint closely maintain ratios 5.00/5.00/5.00 whatever values of the other parameters such as potential demand density, sensitivity factors or speeds. This result is similar to those found in Kocur and Hendrickson (12), and Chang and Schonfeld (20).

The profit and social welfare functions are relatively flat near optimum. For practical applications, this implies that a near optimal profit or welfare can be attained while fitting the transit network to the particular street network or modifying its operating schedule.

The results of maximization of social welfare for different subsidy levels indicate that the welfare function is relatively flat near optimum. A practical application of this result is that for a given set of data input the subsidy can be reduced (or eliminated) by providing passengers a service with marginally worse quality. Therefore, the welfare objective with break-even constraint seems quite reasonable and far more desirable from an economical stand point than the welfare objective with unconstrained subsidy. Furthermore, for a given set of input data in the numerical example, a negative subsidy (i.e., profit) can be obtained for a marginal decrease in social welfare.

**FUTURE RESEARCH**

Several simplifying assumptions should be relaxed in future models. The linear demand function may be replaced by nonlinear function that more precisely reflect the traveler behavior. More realistic and irregular distributions of demand over space (e.g., non-uniform lateral distributions) and time should be used. The model should be improved to handle non-CBD trips (e.g., many-to-many travel pattern) and access modes other than walking. Average stop spacing, assumed to be constant in this research, needs to be optimized. The impacts of demand on operator cost needs to be considered. For example the number of people waiting on the stop to board the vehicle may impact the dwell time and cruising speeds and thus the cost of operation. The cost of transit facilities (e.g., station cost) should be considered in order to make the methodology more applicable in planning fixed guideway modes.
REFERENCES


NOTATION
The following symbols are used in this paper:

\( C_0 \) = operator cost
\( c \) = vehicle operating cost
\( E \) = corridor length
\( g \) = average passenger access speed
\( H \) = operating headway for a transit route
\( K \) = vehicle capacity
\( L \) = length of transit route
\( l \) = allowable peak hour load factor at CBD
\( M \) = lateral route spacing for rectangular area
\( n \) = number of routes
\( P \) = passenger trip density
\( S \) = average stop spacing
\( V \) = average transit operating speed
\( Y \) = corridor width
\( Q \) = total corridor transit demand
\( e_w \) = sensitivity factor for wait time
\( e_a \) = sensitivity factor for access time
\( e_{iv} \) = sensitivity factor for in vehicle time
\( e_p \) = sensitivity factor for fare
\( f \) = fare
\( W \) = social welfare
\( T \) = consumer surplus
\( K \) = subsidy

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Units</th>
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<tr>
<td>( C_0 )</td>
<td>$/hour</td>
</tr>
<tr>
<td>( c )</td>
<td>$/vehicle-hour</td>
</tr>
<tr>
<td>( E )</td>
<td>km</td>
</tr>
<tr>
<td>( g )</td>
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</tr>
<tr>
<td>( H )</td>
<td>hours/vehicle</td>
</tr>
<tr>
<td>( K )</td>
<td>seats/vehicle</td>
</tr>
<tr>
<td>( L )</td>
<td>km</td>
</tr>
<tr>
<td>( l )</td>
<td>km</td>
</tr>
<tr>
<td>( M )</td>
<td>km/route</td>
</tr>
<tr>
<td>( n )</td>
<td></td>
</tr>
<tr>
<td>( P )</td>
<td>passengers/km^2-hour</td>
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<tr>
<td>( S )</td>
<td>km/stop</td>
</tr>
<tr>
<td>( V )</td>
<td>km/hour</td>
</tr>
<tr>
<td>( Y )</td>
<td>km</td>
</tr>
<tr>
<td>( Q )</td>
<td>passengers/hr</td>
</tr>
<tr>
<td>( e_w )</td>
<td></td>
</tr>
<tr>
<td>( e_a )</td>
<td></td>
</tr>
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<td>( e_{iv} )</td>
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<td>( e_p )</td>
<td></td>
</tr>
<tr>
<td>( f )</td>
<td>$/passenger</td>
</tr>
<tr>
<td>( W )</td>
<td>$/hour</td>
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<tr>
<td>( T )</td>
<td>$/hour</td>
</tr>
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<td>( K )</td>
<td>$/hour</td>
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### TABLE 1 PERTINENT ANALYTICAL MODELS FOR TRANSIT SERVICE DESIGN

<table>
<thead>
<tr>
<th>Decision Variables</th>
<th>Objective Function</th>
<th>Transit Mode</th>
<th>Street Network Geometry</th>
<th>Passenger Demand</th>
<th>Authors</th>
</tr>
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<td>Route Length, Spacing, Headway, Stop Spacing</td>
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<td>rectangular and sectorial grid</td>
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<td>Spasovic and Schonfeld (1993)</td>
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<td>Route Length</td>
<td>Min. operator and user cost</td>
<td>rail</td>
<td>rectangular grid</td>
<td>Uniform, inelastic, many-to-one</td>
<td>Wirasinghe and Seneviratne (1986)</td>
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<tr>
<td>Zone Length, Headway</td>
<td>Min. operator and user cost</td>
<td>bus</td>
<td>rectangular grid</td>
<td>Piecewise uniform, inelastic,</td>
<td></td>
</tr>
<tr>
<td>Route Spacing, Lengths and Headway</td>
<td>Min. operator and user cost</td>
<td>bus and rail</td>
<td>rectangular grid</td>
<td>Uniform, inelastic, many-to-one</td>
<td>Byrne (1976)</td>
</tr>
<tr>
<td>Route Spacing</td>
<td>Min. operator and user cost</td>
<td>bus</td>
<td>rectangular grid</td>
<td>Uniform, inelastic, many-to-many</td>
<td>Holroyd (1967)</td>
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<td>Route Spacing and Headway</td>
<td>Min. operator and user cost</td>
<td>bus</td>
<td>rectangular grid</td>
<td>Uniform, inelastic, many-to-one</td>
<td>Byrne and Vuchic (1972)</td>
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<td>Route Density and Frequency</td>
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<td>rectangular grid</td>
<td>General linear, inelastic, many-to-one</td>
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<td>Route Spacing, Headway and Fare</td>
<td>Max. operator profit, Max. user benefit, etc.</td>
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<td>Kocur and Hendrickson (1982)</td>
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<td>Route Spacing, Headway and Stop Spacing</td>
<td>Min. operator and user cost</td>
<td>feeder bus to rail</td>
<td>rectangular grid</td>
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<td>Kuah and Perl (1988)</td>
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<td>Route Spacing, Headway and Fare</td>
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<td>Irregular, elastic, many-to-many, time dependent</td>
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<td>Chang and Schonfeld (1992)</td>
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<td>Station Location and Spacing</td>
<td>Min. total user travel time</td>
<td>rail</td>
<td>linear</td>
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<td>Vuchic and Newell (1968)</td>
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<tr>
<td>Stop Location and Spacing</td>
<td>Min. operator and user cost</td>
<td>rail</td>
<td>rectangular grid</td>
<td>Uniform, inelastic, many-to-one</td>
<td>Hurdle and Wirasinghe (1980)</td>
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### TABLE 2 OPTIMAL OBJECTIVES AND DESIGN VARIABLES

#### A) WITHOUT CAPACITY CONSTRAINT

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<thead>
<tr>
<th></th>
<th>Objective Functions</th>
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<tr>
<td></td>
<td>Profit</td>
<td>Social Welfare</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Unconstrained</td>
<td>Break-Even</td>
<td></td>
</tr>
<tr>
<td>Route Length (km)</td>
<td>7.12 (4.426 miles)</td>
<td>7.16 (4.453 miles)</td>
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<tr>
<td>Route Spacing (km)</td>
<td>2.5 (1.556 miles)</td>
<td>1.94 (1.209 miles)</td>
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<td>Headway (hr)</td>
<td>0.311</td>
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<td>331.397</td>
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<tr>
<td>Profit ($/hr)</td>
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<td>Consumer Surplus ($/hr)</td>
<td>343.747</td>
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<td>Welfare ($/hr)</td>
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<td>Bus Load (pass/bus)</td>
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<td>Fleet Size (buses)</td>
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<td>Marginal Cost for Vehicle Capacity ($/pass)</td>
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#### B) WITH CAPACITY CONSTRAINT

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<tr>
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<td>Profit</td>
<td>Social Welfare</td>
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<td>Unconstrained</td>
<td>Break-Even</td>
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<tr>
<td>Route Length (km)</td>
<td>5.3 (3.296 miles)</td>
<td>4.56 (2.834 miles)</td>
<td>4.56 (2.838 miles)</td>
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<td>Route Spacing (km)</td>
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<td>1.12 (0.699 miles)</td>
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<td>Headway (hr)</td>
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<tr>
<td>Fare ($)</td>
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<td>Operator Cost ($/hr)</td>
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<tr>
<td>Profit ($/hr)</td>
<td>264.242</td>
<td>-150.264</td>
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<tr>
<td>Consumer Surplus ($/hr)</td>
<td>236.772</td>
<td>869.276</td>
<td>710.525</td>
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<tr>
<td>Welfare ($/hr)</td>
<td>501.014</td>
<td>719.012</td>
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<td>Bus Load (pass/bus)</td>
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<td>50</td>
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<td>Fleet Size (buses)</td>
<td>3.279</td>
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<td>Marginal Cost of Subsidy Increase</td>
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### TABLE 3. SENSITIVITY ANALYSIS FOR PROFIT MAXIMIZATION

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Length (miles)</th>
<th>Spacing (miles)</th>
<th>Headway (hr/veh)</th>
<th>Fare ($/pass)</th>
<th>Profit ($/hr)</th>
<th>Demand (pass/hr)</th>
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</thead>
<tbody>
<tr>
<td>Corridor</td>
<td>4.5</td>
<td>2.959</td>
<td>1.017</td>
<td>0.203</td>
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<td>($e_w, e_a$)</td>
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<td>2.976</td>
<td>0.978</td>
<td>0.196</td>
<td>0.88</td>
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<tr>
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## TABLE 4 SENSITIVITY ANALYSIS FOR WELFARE MAXIMIZATION

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<th>Length (miles)</th>
<th>Spacing (miles)</th>
<th>Headway (hr/veh)</th>
<th>Fare ($/pass)</th>
<th>Welfare ($/hr)</th>
<th>Demand (pass/hr)</th>
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</thead>
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<td>Corridor</td>
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<td>2.839</td>
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<td>3.044</td>
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<td>2.591</td>
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<td>3.016</td>
<td>0.726</td>
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<td>0.438</td>
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<td>0.739</td>
<td>0.148</td>
<td>0.454</td>
<td>0.454</td>
<td>710.525</td>
</tr>
</tbody>
</table>
FIGURE 1 METHODOLOGICAL FRAMEWORK FOR OPTIMAL TRANSIT SERVICE COVERAGE

Evaluation Criteria
(e.g., maximize operator profit, social welfare)
FIGURE 2 CORRIDOR AND TRANSIT NETWORK UNDER STUDY

ZONE 2

ZONE 1

CBD

M/4

S/4

M

Y

L

E-L

E
FIGURE 3 OPTIMIZATION ALGORITHM

Input Initial Data

Compute L*

Compute M*

Compute H*

Compute f*

H*=H(Max)

H*<=H(Max)

Convergence Check

Print Optimal Solution

Search Objective Varying L

Stop
FIGURE 4 OPTIMAL TRANSIT ROUTE CONFIGURATION FOR MAXIMUM OPERATOR PROFIT WITH VEHICLE CAPACITY CONSTRAINT

Input Data:
- transit corridor length: 8.045 km (5 miles)
- transit corridor width: 4.827 km (3 miles)
- average transit speed: 16.09 km/hour (10 miles/hour)
- average access speed: 4.02 km/hour (2.5 miles/hour)
- operator cost: $40/vehicle-hour
- passenger density: 77.35 passengers/km²-hour (200 passengers/mile²-hour)
- transit vehicle capacity: 50 seats/vehicle
- allowable peak load factor: 1.0
- stop spacing: 0.402 km/stop (0.25 mile/stop)
- $e_w, e_a : 0.7$
- $e_{iv} : 0.35$
- $e_p : 0.5$

Optimal Values:
- Route Length $L$: 5.23 km (3.296 miles)
- Headway $H$: 0.201 hours
- Spacing $M$: 1.614 km (1.004 miles)
- Fare $f$: $0.883
FIGURE 5. OPTIMAL TRANSIT ROUTE CONFIGURATION FOR MAXIMUM WELFARE WITH VEHICLE CAPACITY CONSTRAINT --With Subsidy

Optimal Values:
Route Length $L$: 4.557 km (2.834 miles)
Headway $H$: 0.140 hours
Spacing $M$: 1.124 km (0.699 miles)
Fare $f$: $0.356
FIGURE 6 OPTIMAL TRANSIT ROUTE CONFIGURATION FOR MAXIMUM WELFARE AND VEHICLE CAPACITY CONSTRAINT - No Subsidy (Break Even)

Optimal Values:

Route Length $L$: 4.551 km (2.83 miles)
Headway $H$: 0.148 hours
Spacing $M$: 1.188 km (0.739 miles)
Fare $f$: $0.454
FIGURE 7. ROUTE LENGTH VS PROFIT

FIGURE 8 ROUTE LENGTH VS SOCIAL WELFARE.
FIGURE 9. SOCIAL WELFARE USER BENEFIT FOR VARIOUS SUBSIDY LEVELS

A)

B)
Appendix A

1. 
\[
L^* = \frac{[2(AB + CD) - A^2 E]}{3(A^2 - C^2)} - \frac{1}{2} \{12ce_p(A^2 - C^2) + HMVP \left[ (AB + CD)^2 + 3(AD + BC)^2 \right] + EHMVP (A^4E + 2A^3B - 4A^2CD - 6ABC^2) \}^{\frac{1}{2}} / (C^2 - A^2)\sqrt{HMVP}
\]

where:
\[
A = \frac{e_a}{2g} - \frac{e_v}{V}
\]
\[
B = 1 - e_w \frac{H}{2} - e_a \frac{M}{4g} - e_a \frac{E}{2g}
\]
\[
C = \frac{e_v}{2V}
\]
\[
D = 1 - e_w \frac{H}{2} - e_a \frac{M + S}{4g}
\]

2. 
\[
L^* = R - \frac{1}{2} \{S(1 + \lambda) + T + fHMVP e_p^2(2(A + B)[B(A + 3C) + D(3A + C)](1 + \lambda) + fEHMVPe_p^2A(A - 3C)(A + C) + f^2HMVP e_p^2 A^2(1 + \lambda)^2 \}^{\frac{1}{2}} / (C^2 - A^2)\sqrt{HMVP}
\]

where:
\[
R = \frac{[2(AB + CD) - A^2 E]}{3(A^2 - C^2)}
\]
\[
S = 12ce_p(A^2 - C^2)
\]
\[
T = HMVP \left[ (AB + CD)^2 + 3(AD + BC)^2 \right]
\]
\[
+ EHMVPA (A^4E + 2A^3B - 4ACD - 6BC^2)
\]
\[
A = \frac{e_a}{2g} - \frac{e_v}{V}
\]
\[
B = 1 - e_w \frac{H}{2} - e_a \frac{M}{4g} - e_a \frac{E}{2g}
\]
\[
C = \frac{e_v}{2V}
\]
\[
D = 1 - e_w \frac{H}{2} - e_a \frac{M + S}{4g}
\]
3. 

\[ H^* = \frac{\sqrt{J}}{\sqrt{A + B(1 - C - D + F - 2G) + L(1 - G - I)}} \]

and

\[ H^* = \frac{\sqrt{J(1 + \lambda)}}{\sqrt{A + B(1 - C - D + F - 2G) + L(1 - G - I) + e_{ef}(-B + (E - L) + E\lambda)}} \]

where:

\[ A = -Ee, / 2 \quad F = e_L / 2g \]
\[ B = E - L \quad G = e_oL / 2V \]
\[ C = e_aM / 4g \quad I = e_o(M + S) / 4g \]
\[ D = e_aE / 2g \quad J = (4ce_pL) / (MVPe) \]

4. 

\[ M^* = \frac{\sqrt{J}}{\sqrt{A + B(1 - C - D + F - 2G) + L(1 - G - I)}} \]

and

\[ M^* = \frac{\sqrt{J(1 + \lambda)}}{\sqrt{A + B(1 - C - D + F - 2G) + L(1 - G - I) + e_{ef}(-B + (E - L) + E\lambda)}} \]

where:

\[ A = -Eea / 4 \]
\[ B = E - L \]
\[ C = e_\theta H / 2 \]
\[ D = eaE / 2g \]
\[ F = eaL / 2g \]
\[ G = e_vL / 2V \]
\[ I = ea(2Hg + S) / 4g \]
\[ J = (8cgepL) / (HVPe) \]

5. 

\[ \hat{\lambda} = \frac{-X + \sqrt{X^2 - 4(A - B)(X + ICDJ + IFGJ - Z)}}{2(X + ICDJ + IFGJ - Z)} \]
where:

\[A = 2cLY / HMV\]
\[C = (E - L)PY\]
\[D = 1 - e_a(E - L)/2g - e_aM/4g - Le_w/V - He_w/2\]
\[F = LPY\]
\[G = 1 - e_a(M + S)/4g - Le_w/2V - He_w/2\]
\[J = -4ELge_w + 2L^2ge_w - 2E^2Ve_a + 4EVg + 4ELVe_a - 2e_aL^2V - e_aEMV - e_aLSV - 2EgHV e\]
\[I = 4EgPV\]

6.

\[H^* = C^2D - E + AE + BE + Q + FL + GL\]
\[+ \sqrt{((-C^2D + E - AE - BE - Q - FL - GL)^2 + 4XZ)}\]

where:

\[Z = -e_aE/2\]
\[A = e_aM/4g\]
\[B = e_a\]
\[C = E - L\]
\[D = e_a/2g\]
\[E = e_a(E - L)\]
\[F = e_aS/4g\]
\[G = e_aL/2V\]
\[X = \frac{cap}{PM}\]
Finally, the model with profit maximizing objective can be written in the following form:

Maximize $\Pi(L, H, M, f) = PYE [1 - e_e H/2 - e_a M/4g - e_e f] f + PY(E - L)[-e_a (E - L)/2g - e_e L/V] f + PYL[-e_v S/4g - e_v L/2V] f - 2cYL/\text{HMV}$

subject to:

$PYE [1 - e_e H/2 - e_a M/4g - e_e f] + PY(E - L)[-e_a (E - L)/2g - e_e L/V] + PYL[-e_v S/4g - e_v L/2V] K Y/\text{MH} l$

$L, H, M, f \geq 0$

(o) In welfare maximization with capacity constraint break even case, the capacity constraint is not binding (bus load = 49.267). Why in the break even case without capacity constraint we have different results?